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Large-scale instabilities of poloidal magnetic fields in stars

por

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Abstract

In 1977, Flowers and Ruderman described a perturbation that destabilized a purely dipolar magnetic field in a star. They considered the effect of cutting the star in half along a plane parallel to the symmetry axis and rotating each half 90 degrees in opposite directions, which would cause the energy of the magnetic field in the exterior of the star to be greatly reduced, just as it happens with a pair of aligned magnets. We formally solve for the energy of the external magnetic field and check that it decreases monotonously along the entire rotation. We also describe the instability using perturbation theory, and see that it happens due to the work done by the interaction of the magnetic field with surface currents. Finally, we consider the stabilizing effect of adding a toroidal field by studying the internal energy perturbation when the rotation is not done along a sharp cut, but with a continuous displacement field that switches the direction of rotation across a region of small but finite width. Using these results, we estimate the relative strengths of the toroidal and poloidal field needed to make the star stable to this displacement and see that the energy of the toroidal field required for this is much smaller than the energy of the poloidal field.

Contents

Introduction

Large-scale magnetic fields are known to be present in a wide variety of stellar objects, meaning that in these stars the dipole component (together perhaps with some other low-order multipoles) is not much weaker than the rms surface field. The initial discovery of such fields was on Ap stars (Babcock 1947). Since then, they have been observed or inferred to exist in white dwarfs, neutron stars, upper-main-sequence stars, and in the central stars of planetary nebulae (e.g. Kemp et al. (1970), Angel et al. (1974), Henrichs et al. (2003), Jordan et al. (2005)). Also, these fields appear to be long-lived, since they do not evolve in a timescale accesible to observations.

A common feature of all these objects is that, over most of their structure, they are stably stratified. White dwarfs and neutron stars have no significant convective regions[1], while upper-main-sequence stars only have a small convective core. Dynamo effects are therefore expected to be irrelevant in keeping the strength of the magnetic field constant. Also, it can be seen that all these objects have very similar magnetic fluxes on their surfaces, $\Phi_{max} = \pi R^2 B_{max} \sim 10^{27.5}$ G cm^2, where B_{max} is the highest surface dipole strength detected each class of objects. These two features are considered compelling arguments in favor of flux freezing during stellar evolution. Also, it can be seen that the ratio of fluid to magnetic pressure is (Reisenegger 2009)

$$\beta = \frac{8\pi P}{B^2} \sim \frac{8\pi^3 GM^2}{\Phi^2} \sim 3 \times 10^6 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\Phi}{\Phi_{max}}\right)^{-2}, \qquad (1)$$

which is a very high number even for the most strongly magnetized stars. Also, β is similar for all the objects mentioned. Since this ratio is so high, we do not expect these fields to significantly modify the

[1]Recently formed neutron stars are convective for some seconds, and white dwarfs have a thin convective region on their surface.

structure of the star. However, they can play a major role in its evolution.

Even though these long-lived fields have been known to exist for more than half a century, it has not been possible to find an analytic model for a field that is in a stable equilibrium. However, stable configurations have been found to exist via numerical calculations (Braithwaite & Nordlund 2006), where an initially random field usually evolves into an approximately axisymmetric configuration that is a combination of toroidal and poloidal components of similar energies. In these simulations, once a stable configuration has been achieved, the decay of the field is driven by Ohmic dissipation, and it can be seen to evolve in a timescale comparable to the lifetime of the star.

The stability of purely poloidal or purely toroidal fields has also been studied in the past. Tayler (1973), using the energy method, proved that every purely toroidal field is unstable on an Alfvén timescale, independent of the strength of the field. Markey & Tayler (1973, 1974) and independently Wright (1973) discovered that purely poloidal fields with closed lines contained inside the star are also unstable. These instabilities are very similar to the kink instabilities in a z-pinch. For toroidal fields, the region close to the symmetry axis resembles this type of pinch, and the same occurs near the magnetic axis of poloidal fields. The fundamental difference with a kink instability is the restriction that in stably stratified stars displacements perpendicular to equipotential surfaces are unlikely to be unstable, since the magnetic pressure is significantly smaller than the fluid pressure. For the same reason, it is expected that unstable displacements $\boldsymbol{\xi}$ are nearly incompressible (i.e. $\nabla \cdot \boldsymbol{\xi} \simeq 0$).

A simple argument given by Flowers & Ruderman (1977) shows that any poloidal field with field lines extending outside the star should be unstable. If the initial configuration is such that the external field resembles a dipole, cutting the star in half and rotating each half by 90 degrees in opposite directions would greatly reduce the dipole component of the field, leading to a magnetic field with less energy. However, neither Flowers and Ruderman nor anyone else has ever given a formal proof of this argument.

In the numerical simulations of Braithwaite (2007, 2009), instabilities related to the poloidal field are studied. In the latter work mentioned, using the stable configurations found after simulating the evolution of random fields, Braithwaite used different ratios of poloidal to total energy of the magnetic field E_P/E and saw the field to be stable for E_P/E smaller than 0.8 but larger than 0.056. The field became unstable for E_P/E greater than 0.8, with an $m = 2$ mode that seems to consist mostly of displacements in latitude of the fluid. Braithwaite also notes that the $m = 0$ and $m = 1$ modes are not physically possible, since they would break the conservation of linear and angular momentum, respectively. At ratios over $E_P/E = 0.9$, higher modes became unstable, as would be expected since these modes have to overcome a higher resistance from the toroidal field. Braithwaite (2007) considers the stability of purely poloidal fields with closed field lines inside the star, and found them to be unstable with a mode much higher than $m = 2$. The displacement in both cases resembles a kink instability, as mentioned above.

In this work, I formally prove the Flowers and Ruderman's instability for the case in which the external field is that of a pure dipole. I then study the stabilizing effect of a toroidal field and the relative strength of toroidal and poloidal components required to stabilize the star against Flowers and Ruderman's instability.

The structure of this report is the following: In Chapter 1, I formally prove Flowers & Ruderman's instability for a pure dipole field, by solving exactly the energy of the external magnetic field during the entire process. Using perturbation theory, I see that the instability is caused by the effect of surface currents on the star. In Chapter 2, I consider the stabilizing effect of a toroidal field when the perturbation is not done with a sharp cut through the star, but rather with a displacement field that switches continuously from one direction of rotation to the other, over a thin but finite region. I show than under some reasonable assumptions the energy of the toroidal magnetic field required to stabilize the star is much smaller than that of the poloidal field. In chapter 3, I present my general conclusions and discuss ongoing work in which I try to construct a displacement field in order to reproduce the results obtained by Braithwaite (2009).

Chapter 1

Flowers & Ruderman's instability for a pure dipole field

Flowers & Ruderman (1977) claimed that stars with purely poloidal fields were unstable to a fluid displacement where the star was cut in half and each piece was rotated 90 degrees in opposite directions, generating a quadrupole, which should have less energy than the initial configuration. The argument was given as an analogy with two aligned magnets, in which case, the antiparallel configuration has less energy. However, the star in its interior preserves the magnitude but changes the direction of the field, while the magnets preserve their magnetization, and the field along them does not have a constant magnitude.

In this chapter, I present a formal proof of Flowers & Ruderman's instability for the case in which the external field is that of a point dipole. In §1.1 I provide a formal proof of the instability by exactly solving the energy of the external field along the entire rotation. I start in §1.1.1, by obtaining a general expression for the external magnetic energy for an arbitrary field. In §1.1.2, I prove that under certain conditions that are valid for the displacement done in Flowers & Ruderman's instability, the final state of the star will have less energy than the initial one, as long as the initial configuration of the external field is a pure dipole. In §1.1.3, I use the expression for the external magnetic energy obtained in §1.1.1 to complete the proof by showing that the external energy of the field is a monotonous function of the angle of rotation, which, coupled with the result of §1.1.2, completes the proof of Flowers & Ruderman's instability. In §1.2, I demonstrate how this instability can be understood using perturbation theory. Although the proof using perturbation theory will not be as complete as the one given in §1.1, the results obtained are useful for the work described in the following Chapters.

1.1 Proof of Flowers and Ruderman's instability by an exact evaluation of the energy

If I completely ignore the effects of the magnetic field over the hydrostatic structure of the star, then the star should be perfectly spherical, and when Flowers and Ruderman's instability takes place, each half of the star rotates as a rigid solid. Since in stellar interiors the magnetic Reynolds number is significantly larger than 1, field lines will be dragged by the fluid without modyfing the magnitude of the magnetic field at each point, and thus, the internal magnetic energy of the star will not be modified in the process[1]. Therefore, we are only interested then in the energy of the external magnetic field, and I now proceed to prove that this energy is in fact reduced by performing the displacement described in Flowers and Ruderman's instability.

1.1.1 Exterior energy of an arbitrary magnetic field

To start, I must obtain the magnetic field outside the star, given the field on its surface. Because outside the star there are no currents, we have $\nabla \times \boldsymbol{B} = 0$ and therefore $\boldsymbol{B} = \nabla\Psi$. Since $\nabla \cdot \boldsymbol{B} = 0$, Ψ must satisfy Laplace's equation. The general solution to Laplace's equation in spherical coordinates is given by

$$\Psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right]Y_{lm}(\theta,\phi). \tag{1.1}$$

All the coefficients a_{lm} must be equal to zero since Ψ must tend to zero as r goes to infinity, thus

$$\Psi(r,\theta,\phi) = \sum_{lm}\frac{b_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi). \tag{1.2}$$

[1] It is important to note that the plane that cuts the star cannot cross any field lines, otherwise, to rotate each half these field lines should be cut, and that is not possible.

The values for b_{lm} can be solved because the component of the magnetic field normal to the surface of the star must be continuous, so

$$\left(\frac{\partial \Psi}{\partial r}\right)_{r=R} = (\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R} \tag{1.3}$$

$$-\sum_{lm}(l+1)\frac{b_{lm}}{R^{l+2}}Y_{lm}(\theta,\phi) = (\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}, \tag{1.4}$$

and using the orthonormality of spherical harmonics, I can get

$$b_{lm} = -\frac{R^{l+2}}{l+1}\int_{4\pi} \mathrm{d}\,\Omega(\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}Y^*_{lm}(\theta,\phi). \tag{1.5}$$

The final result is clearer if I define coefficients c_{lm} such that

$$b_{lm} = -\frac{R^{l+2}}{l+1}c_{lm} \quad \Longrightarrow \quad \Psi(r,\theta,\phi) = -\sum_{lm}\frac{R^{l+2}c_{lm}}{r^{l+1}(l+1)}Y_{lm}(\theta,\phi). \tag{1.6}$$

Now, the magnetic energy inside the star should not change, since the field only rotates while keeping its magnitude. However, the exterior field changes significantly. Thus, the variation of the magnetic energy can be solved just by solving the variation outside of the star. The exterior magnetic energy is obtained from

$$E = \int_V \mathrm{d}\,V\frac{B^2}{8\pi} = \int_V \mathrm{d}\,V\frac{(\nabla\Psi)^2}{8\pi} = \frac{1}{8\pi}\left[\int_V \mathrm{d}\,V\nabla\cdot(\Psi\nabla\Psi) - \int_V \mathrm{d}\,V\Psi\nabla^2\Psi\right] \tag{1.7}$$

$$= \frac{1}{8\pi}\int_V \mathrm{d}\,V\nabla\cdot(\Psi\nabla\Psi) \tag{1.8}$$

where V covers all space outside the star. Using the divergence theorem, the energy can be expressed as a surface integral, with a normal inward to the star:[2]

$$E = \frac{1}{8\pi}\oint_S(\Psi\nabla\Psi)_{r=R}\cdot\,\mathrm{d}\,\boldsymbol{s}. \tag{1.9}$$

[2]The surface of integration in this case consists in the surface of the star, plus a surface at infinity. If I consider this last surface as a sphere of radius r centered at the star, then its area goes like r^2. From Ψ, it can be seen that the term that decreases less significantly with r goes like r^{-1}, and thus, the corresponding terms for $\nabla\Psi$ will go like r^{-2}. So, in the limit in which $r \to \infty$, the integral corresponding to this surface will go like r^{-1}, so when I take the surface at infinity, this term does not contribute, and I only need to consider the term with the surface of the star.

Since I consider the star to be perfectly spherical, $\nabla\Psi \cdot \mathrm{d}\,\boldsymbol{s} = -R^2(\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\sin\theta\,\mathrm{d}\,\theta\,\mathrm{d}\,\phi$ on the surface of the star, and consequently

$$E = -\frac{R^2}{8\pi}\int_{4\pi}(\Psi\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\,\mathrm{d}\,\Omega = -\frac{R^2}{8\pi}\int_{4\pi}(\Psi^*\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\,\mathrm{d}\,\Omega \tag{1.10}$$

where in the last step, I used the fact that the energy is real and set it equal to its conjugate. Replacing the expression for Φ, I get the following result:

$$E = \frac{R^3}{8\pi}\sum_{lm}\int_{4\pi}\frac{c_{lm}^*}{l+1}Y_{lm}^*(\theta,\phi)(\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\,\mathrm{d}\,\Omega = \frac{R^3}{8\pi}\sum_{lm}\frac{|c_{lm}|^2}{l+1}. \tag{1.11}$$

Summing up, the energy of the external magnetic field is

$$\boxed{E = \frac{R^3}{8\pi}\sum_{lm}\frac{|c_{lm}|^2}{l+1} \qquad c_{lm} = \int_{4\pi}Y_{lm}^*(\theta,\phi)(\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\,\mathrm{d}\,\Omega}. \tag{1.12}$$

1.1.2 Proof that the final energy is less than the initial one

The results contained in (1.12) are enough to obtain a formal proof of Flowers & Ruderman's instability. To do so, let us define a quantity Υ as

$$\Upsilon = \frac{R^3}{8\pi}\int_{4\pi}\left(B_r^2\right)_{r=R}\,\mathrm{d}\,\Omega. \tag{1.13}$$

This quantity will be conserved when the star is cut in half and rotated. So, using the superscripts i and f to denote initial and final states, $\Upsilon^i = \Upsilon^f$. If I use the spherical harmonics expansion for the field outside the star to express one of the terms of $\left(B_r^2\right)_{r=R}$, I get

$$\Upsilon = \frac{R^3}{8\pi}\int_{4\pi}(B_r)_{r=R}\left[\sum_{lm}c_{lm}Y_{lm}(\theta,\phi)\right]\mathrm{d}\,\Omega \tag{1.14}$$

$$= \frac{R^3}{8\pi}\sum_{lm}c_{lm}\int_{4\pi}(B_r)_{r=R}Y_{lm}(\theta,\phi)\,\mathrm{d}\,\Omega \tag{1.15}$$

$$= \frac{R^3}{8\pi}\sum_{lm}|c_{lm}|^2. \tag{1.16}$$

By rewriting $\Upsilon^i = \Upsilon^f$, I obtain

$$\frac{R^3}{8\pi}\sum_{lm}|c^f_{lm}|^2 = \frac{R^3}{8\pi}\sum_{lm}|c^i_{lm}|^2. \tag{1.17}$$

If the initial external field is that of a point dipole, the only nonzero c^i_{lm} is c^i_{10}. Considering this, dividing the above expression by 2 results in

$$\frac{R^3}{8\pi}\sum_{lm}\frac{|c^f_{lm}|^2}{2} = \frac{R^3}{8\pi}\frac{|c^i_{10}|^2}{2}. \tag{1.18}$$

From here, using (1.12) and noting that c_{00} must be equal to zero both in the initial and final state because it represents a monopole, I get:

$$E_f \leq \frac{R^3}{8\pi}\sum_{lm}\frac{|c^f_{lm}|^2}{2} = \frac{R^3}{8\pi}\frac{|c^i_{10}|^2}{2} = E_i. \tag{1.19}$$

Thus, the final state will have less or equal energy than the initial one. The equality will hold if and only if the c^f_{lm} are equal to zero when $l \neq 1$, which is not the case in Flowers & Ruderman's instability since the severe discontinuity that is produced cannot be resolved into an expansion of spherical harmonics with a finite number of terms. The result given in (1.19) not only holds for Flowers & Ruderman's instability, but for any perturbation that keeps Υ constant. Perhaps studying the conditions that the displacement field must satisfy in order for Υ to be kept constant might allow us to discover other interesting instabilities that affect poloidal fields, but I will not deal with that problem in this work.

It is important to note, however, that I only proved that the magnetic energy of any final state after cutting the star and rotating it is less than that of the initial energy of the dipole field. I have yet to prove that the energy is monotonously decreasing for the entire rotation. So, up to this point, we could expect the minimun energy to be present at some intermediate point in the rotation, and not after the rotation has been completed.

1.1.3 Proof that the energy decreases monotonously

The boundary condition on the magnetic field required on the surface of the star, so the external field is a pure dipole, is that the radial component satisfy $(B_r)_{r=R} = B_P \cos\theta$, where B_P is the strength of the field on the surface exactly at the symmetry axis. Since the internal field is irrelevant to the problem (as long as no field lines are cut during the process), I will consider a uniform magnetic field $\boldsymbol{B} = B_P \hat{\boldsymbol{z}}$ inside the star. This field satisfies the required boundary condition, and proving Flowers & Ruderman's instability for it will be enough as a proof for any other field that is a pure dipole in the exterior. Since the external field in this case is that of a point dipole, the final energy must be smaller than the initial one, as I proved in the previous section.

If the star is cut in half along a plane perpendicular to the x axis, and each half of the star is rotated by an angle Ω in opposite directions, I get

$$\boldsymbol{B} = \begin{cases} B_P(\cos\Omega \hat{\boldsymbol{z}} - \sin\Omega \hat{\boldsymbol{y}}) & x > 0 \\ B_P(\cos\Omega \hat{\boldsymbol{z}} + \sin\Omega \hat{\boldsymbol{y}}) & x < 0 \end{cases} \tag{1.20}$$

for the field inside the star. Its radial component is[3]

$$\boldsymbol{B} \cdot \hat{\boldsymbol{r}} = \begin{cases} B_P(\cos\Omega \cos\theta - \sin\Omega \sin\theta \sin\phi) & x > 0 \\ B_P(\cos\Omega \cos\theta + \sin\Omega \sin\theta \sin\phi) & x < 0 \end{cases} \tag{1.21}$$

Using (1.12), the external magnetic energy corresponding to the unperturbed state can be evaluated as

$$E_0 = \frac{B_P^2 R^3}{12}. \tag{1.22}$$

For the rotated case, the c_{lm} are

$$c_{lm} = \cos\Omega B_P \int_{4\pi} \mathrm{d}\,\Omega \cos\theta Y_{10}^* \tag{1.23}$$

$$+ B_P \sin\Omega \left[\int_0^{\pi} \int_{\pi/2}^{3\pi/2} \sin^2\theta \sin\phi Y_{lm}^* \,\mathrm{d}\,\phi \,\mathrm{d}\,\theta - \int_0^{\pi} \int_{-\pi/2}^{\pi/2} \sin^2\theta \sin\phi Y_{lm}^* \,\mathrm{d}\,\phi \,\mathrm{d}\,\theta \right]. \tag{1.24}$$

[3]Here I use the relation between basis vectors

$$\hat{\boldsymbol{r}} = \sin\theta \cos\phi \hat{\boldsymbol{x}} + \sin\theta \sin\phi \hat{\boldsymbol{y}} + \cos\theta \hat{\boldsymbol{z}}$$

Due to the symmetries of Y_{lm}, it can be proved that

$$c_{lm} = \begin{cases} B_P \cos\Omega \int_{4\pi} \mathrm{d}\,\Omega \cos\theta Y_{10}^* & l=1, m=0 \\ 2B_P \sin\Omega \int_0^\pi \mathrm{d}\,\theta \int_{\pi/2}^{3\pi/2} \mathrm{d}\,\phi \sin^2\theta \sin\phi Y_{lm}^* & l, m \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{1.25}$$

Now, if I define $w_{lm} = \int_0^\pi \mathrm{d}\,\theta \int_{\pi/2}^{3\pi/2} \mathrm{d}\,\phi \sin^2\theta \sin\phi Y_{lm}^*$, the energy can be rewritten as:

$$E = \cos^2\Omega E_0 + \sin^2\Omega \frac{R^3 B_P^2}{2\pi} \sum_{\substack{lm \\ \text{even}}} \frac{|w_{lm}|^2}{l+1} = E_0 \left[\cos^2\Omega + \frac{6\sin^2\Omega}{\pi} \sum_{\substack{lm \\ \text{even}}} \frac{|w_{lm}|^2}{l+1}\right]. \tag{1.26}$$

Defining $A = \frac{6}{\pi} \sum_{\substack{lm \\ \text{even}}} \frac{|w_{lm}|^2}{l+1}$, the energy can be rewritten in a compact form as

$$\boxed{E = E_0 \left[1 + \sin^2\Omega(A-1)\right]}. \tag{1.27}$$

Since the complete rotation is obtained with $\Omega = \pi/2$, the energy is a monotonous function of Ω. If $A > 1$ the energy will be an increasing function, but if $A < 1$, it will be a decreasing function. However, we already know that the final energy is smaller than the initial one, thus $A < 1$ and the energy decreases monotonously along the entire rotation. Therefore, Flowers & Ruderman's instability is present in the case of the purely dipolar field.

Even though we already proved the existence of the instability, an estimate of A is called for. To obtain this estimate, I consider the quantity

$$g(l) = \frac{6}{\pi} \sum_{m=-l}^{l} \frac{|w_{lm}|^2}{l+1}. \tag{1.28}$$

A log-log plot for these values is shown in figure 1.1 for l ranging from 6 to 18. A good fit can be achieved by the function $g(l) = 2.5303 \cdot l^{-2.8871}$ (I ignored the values for l=2, 4 because they did not fit well with the rest of the data). I use this power law to estimate the result of the sum in A for the terms

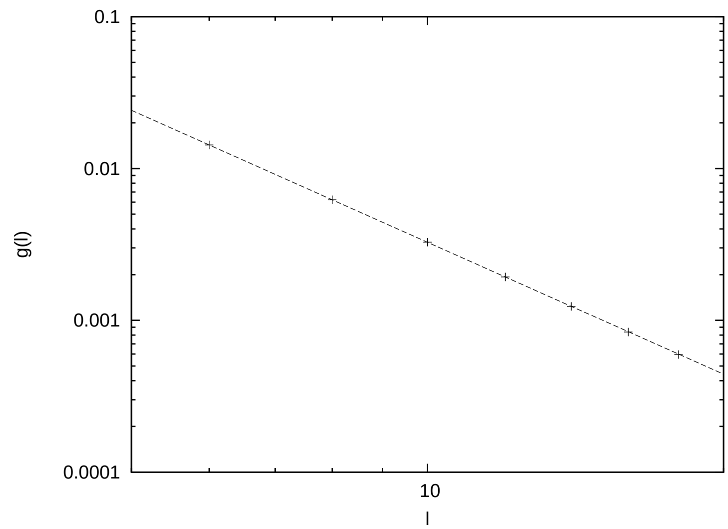

Figure 1.1: Approximation of $g(l)$ by a power law. Only the points from $l = 6$ to 18 are considered because the points $g(2)$ and $g(4)$ don't fit well with the rest of the data. It is found that a good fit is obtained with $g(l) = 2.5303 \cdot l^{-2.8871}$

with $l \geq 6$, and the exact values obtained for the terms with $l = 2$ and $l = 4$, which gives me a value of $A = 0.5697$, while the direct sum of the terms up to $l = 18$ gives me $A = 0.5685$. As expected, this value satisfies the condition $A < 1$ that is required for Flowers & Ruderman's instability to be valid.

1.2 Proof of the instability using perturbation theory

Using MHD perturbation theory, I should also be able to prove the existence of Flowers & Ruderman's instability. This proof however will not be as complete as the one given in 1.1, since perturbation theory can only be used to see if the system is unstable against small displacements, and thus, I cannot prove with this approach that the energy decreases monotonously along the entire rotation. Nevertheless, I now provide a proof of the instability using perturbation theory, since the results obtained in doing so will be useful in the next Chapter.

In §1.2.1, I prove that for a certain family of axisymmetric magnetic fields for which the external field is that of a dipole, the volume contribution to the internal energy perturbation is equal to zero. In §1.2.2, I solve the contribution to the internal energy perturbation due to surface currents and show that the final result directly relates with the energy given in (1.27).

1.2.1 Contribution to the internal energy perturbation inside the star

Using the energy principle from Bernstein et al. (1958), the stability of a system perturbed by a displacement field $\boldsymbol{\xi}$ is given by the sign of the potential energy perturbation which can be written as a sum of hydrostatic and magnetic terms

$$\begin{aligned}\delta W =& \delta W_{hyd} + \delta W_{mag},\\ \delta W_{hyd} =& \frac{1}{2}\int\left[\Gamma_1 P(\nabla\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\xi}\cdot\nabla P)(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\Phi)(\nabla\cdot\rho\boldsymbol{\xi}) + \rho\boldsymbol{\xi}\cdot\nabla\delta\Phi\right]\mathrm{d}\,V\\ &-\frac{1}{2}\oint\left[\Gamma_1 P\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla P\right]\boldsymbol{\xi}\cdot\,\mathrm{d}\,\boldsymbol{s},\\ \delta W_{mag} =& -\frac{1}{2}\int_V \boldsymbol{\xi}\cdot(\delta\boldsymbol{j}\times\boldsymbol{B} + \boldsymbol{j}\times\delta\boldsymbol{B})\,\mathrm{d}\,V\end{aligned} \tag{1.29}$$

where V now denotes the volume of the star. In here, P is the fluid pressure, ρ is the mass density, Φ is the gravitational potential, Γ_1 is defined as

$$\Gamma_1 = \frac{\partial\ln P}{\partial\ln\rho} \tag{1.30}$$

and $\boldsymbol{j}$ is the current density, that in the ideal MHD approximation can be solved as

$$4\pi\boldsymbol{j} = \nabla\times\boldsymbol{B}. \tag{1.31}$$

The perturbed magnetic field and current are given by

$$\delta\boldsymbol{B} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}),\;\; 4\pi\delta\boldsymbol{j} = \nabla\times(\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})), \tag{1.32}$$

and I will employ the Cowling approximation of neglecting perturbations of the gravitational potential (i.e. $\delta\Phi = 0$).

If $\delta W < 0$, then the resulting configuration will be unstable. Since in the stellar interior the magnetic pressure is much smaller than the fluid pressure, I expect instabilities driven by the magnetic field to minimize the magnitude of δW_{hyd}. I ignore the effects of the magnetic field on the structure of the star, so P, ρ and Φ are spherically symmetric. Considering this, if I use a displacement field that has no radial component (which is reasonable due to the stable stratification of the objects we are interested in), and is completely incompressible (i.e. $\nabla \cdot \boldsymbol{\xi} = 0$), the fluid contribution is exactly equal to zero, and I only need to consider the magnetic terms in δW.

The displacement field for the case of Flowers & Ruderman's instability is taken to be

$$\boldsymbol{\xi} = \begin{cases} \Omega r \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} = -\Omega r(\cos\theta\cos\phi\hat{\boldsymbol{\phi}} + \sin\phi\hat{\boldsymbol{\theta}}) & x > 0 \\ -\Omega r \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} = \quad \Omega r(\cos\theta\cos\phi\hat{\boldsymbol{\phi}} + \sin\phi\hat{\boldsymbol{\theta}}) & x < 0 \end{cases} \tag{1.33}$$

with $|\Omega| \ll 1$. This displacement field has no radial component and is incompressible, so there will be no fluid contributions to δW.

For the magnetic field, I will consider configurations given by

$$\boldsymbol{B} = \nabla\alpha \times \nabla\phi, \quad \alpha = f(r)\sin^2\theta. \tag{1.34}$$

On the surface of the star, the radial component for these fields is $\dfrac{2f(R)}{R^2}\cos(\theta)$, and thus, outside the star all these fields are pure dipoles. This model for the internal field covers a wide range of axisymmetric configurations; the constant field studied in the previous section is just a particular case in which $f(r) = B_P r^2/2$ and the fields used by Braithwaite (2007) to study the stability of purely poloidal fields are also of this form.

With this choice of $\boldsymbol{\xi}$ and $\boldsymbol{B}$, the integrand of δW_{mag} is found to be:

$$\boldsymbol{\xi} \cdot (\delta \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{j} \times \delta\boldsymbol{B}) = \frac{\Omega^2 f}{2\pi r^2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} - \frac{2f}{r^2}\right)\left[\cos^2\theta - \sin^2\theta\sin^2\phi\right]. \tag{1.35}$$

Including the $\sin\theta$ term from $\mathrm{d}V$ and performing the integral gives zero as a result.

However, we already saw that there is an effective variation of the energy when performing this perturbation, and thus, we are not taking into account all the work that is done on the fluid. This large scale displacement produces surface currents in two different regions, and these are responsible for the work done:

- Along the surface of the sphere. Since the exterior field satisfies Laplace's equation, and its boundary conditions only requiere the normal component of $\boldsymbol{B}$ to be continuous, it is unlikely that a large-scale displacement that affects the surface of the star will not produce a discontinuity of the tangential component of $\boldsymbol{B}$ in some areas. Thus, surface currents are an important element for perturbations that affect the surface.

- Along the plane that cuts the star. The discontinuity produced by the rotation will produce a current sheet along this plane.

From these two effects, only the first is really relevant to the energy of the star. The second effect is not, because $\boldsymbol{\xi}, \boldsymbol{B}$, and $\boldsymbol{j}$ are parallel to that surface, and thus it does no work on the fluid.

1.2.2 Contribution to the internal energy perturbation due to surface currents

Due to the discontinuity of the θ and ϕ components of the magnetic field, a surface current will be produced with components

$$4\pi K_\theta = B_{\phi int} - B_{\phi ext}, \qquad 4\pi K_\phi = B_{\theta ext} - B_{\theta int}. \tag{1.36}$$

If the field is perturbed by a displacement $\boldsymbol{\xi}$, then $\boldsymbol{B}_{int}$ changes to first order in $\boldsymbol{\xi}$ by $\delta\boldsymbol{B}_{int} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_{int})$. This change will modify the boundary conditions for the exterior field, giving rise to a perturbed exterior magnetic field

$$\delta\boldsymbol{B}_{ext} = \nabla\delta\Phi, \tag{1.37}$$

with

$$\delta\Phi = -\sum_{l,m} \frac{R^{l+2}\delta c_{lm}}{r^{l+1}(l+1)} Y_{lm}(\theta,\phi), \qquad \delta c_{lm} = \int_{4\pi} Y^*_{lm}(\theta,\phi)(\delta\boldsymbol{B}\cdot\hat{\boldsymbol{r}})_{r=R}\,\mathrm{d}\,\Omega. \tag{1.38}$$

This will give rise to a perturbed surface current with components

$$4\pi\delta K_\theta = \delta B_{\phi int} - \delta B_{\phi ext}, \quad 4\pi\delta K_\phi = \delta B_{\theta ext} - \delta B_{\theta int}. \tag{1.39}$$

Now, by replacing $\boldsymbol{j}$ by $\boldsymbol{j} + \delta(r-R)\boldsymbol{K}$ in (1.29) and performing the radial integral for the term with the surface current and the one with the perturbed surface current the contribution to δW due to these terms can be written as[4]

$$\delta W_{sc} = -\frac{R^2}{2}\int_{4\pi} \left(\boldsymbol{\xi}\cdot[\boldsymbol{K}\times\delta\boldsymbol{B} + \delta\boldsymbol{K}\times\boldsymbol{B}]\right)_{r=R} \mathrm{d}\,\Omega. \tag{1.40}$$

However, due to the discontinuity of $\boldsymbol{B}$ and $\delta\boldsymbol{B}$ along the boundary, the choice for these two vectors is somewhat ambiguous. This can be avoided by considering only perturbations that are parallel to the surface, so $\boldsymbol{\xi} = \xi_\theta\hat{\boldsymbol{\theta}} + \xi_\phi\hat{\boldsymbol{\phi}}$, in which case only the radial components of $\boldsymbol{B}$ and $\delta\boldsymbol{B}$ contribute to the previous expression, and that expression reduces to

$$\delta W_{sc} = -\frac{R^2}{8\pi}\left[\int_{4\pi}\delta B_r\boldsymbol{\xi}\cdot(\boldsymbol{B}_{ext} - \boldsymbol{B}_{int})\,\mathrm{d}\,\Omega + \int_{4\pi} B_r\boldsymbol{\xi}\cdot(\delta\boldsymbol{B}_{ext} - \delta\boldsymbol{B}_{int})\,\mathrm{d}\,\Omega\right]. \tag{1.41}$$

Here, it is not necessary to distinguish between the interior and exterior values of B_r and δB_r because these must be continuous. The primary difficulty in this expression is the term $\delta\boldsymbol{B}_{ext}$. However, by explicitly writing that term and using integration by parts, it can be seen that[5]

$$-\frac{R^2}{8\pi}\int_{4\pi} B_r\boldsymbol{\xi}\cdot\delta\boldsymbol{B}_{ext}\,\mathrm{d}\,\Omega = \frac{R^3}{8\pi}\sum_{lm}\frac{|\delta c_{lm}|^2}{l+1}, \tag{1.42}$$

so this term is always positive, and thus does not drive the instability. This expression is still hard to solve analytically in most cases. However, given a particular displacement field, it can be used in the same way as I used the sequence $g(l)$ in section 1.1.3 to obtain an estimate of A.

Now I consider the perturbation field given in (1.33) and the magnetic field given by (1.34). In this case, the δc_{lm} are

$$\delta c_{lm} = \begin{cases} 2B_P\Omega\int_0^\pi \mathrm{d}\,\theta\int_{\pi/2}^{3\pi/2}\mathrm{d}\,\phi\sin^2\theta\sin\phi Y_{lm}^* & l, m \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{1.43}$$

[4]Considering this, δW_{mag} now consists on a volume integral and surface integral:

$$\delta W_{mag} = -\frac{1}{2}\int_V \boldsymbol{\xi}\cdot(\delta\boldsymbol{j}\times\boldsymbol{B} + \boldsymbol{j}\times\delta\boldsymbol{B})\,\mathrm{d}\,V - \frac{R^2}{2}\int_{4\pi}\left(\boldsymbol{\xi}\cdot[\boldsymbol{K}\times\delta\boldsymbol{B} + \delta\boldsymbol{K}\times\boldsymbol{B}]\right)_{r=R}\mathrm{d}\,\Omega.$$

[5]This result **does not depend** on the geometry of the magnetic field. The only requirement, is that the displacement field be of the form $\boldsymbol{\xi} = \xi_\theta\hat{\boldsymbol{\theta}} + \xi_\phi\hat{\boldsymbol{\phi}}$ in the surface. Also, if the displacement field has a radial part, $\xi_r\hat{\boldsymbol{r}}$, then $\partial_r\xi_r$ must vanish on the surface.

Using this, together with equation (1.42) I get

$$-\frac{R^2}{8\pi}\int_{4\pi} B_r\boldsymbol{\xi}\cdot\delta\boldsymbol{B}_{ext}\,\mathrm{d}\,\Omega = \Omega^2 E_0 A, \tag{1.44}$$

where A has the same meaning as in §1.1.3. This gives me one of the terms of δW_{sc} (as shown in equation (1.41)). The other terms can be evaluated directly, giving me the result

$$-\frac{R^2}{8\pi}\left[\int_{4\pi}\delta B_r\boldsymbol{\xi}\cdot(\boldsymbol{B}_{ext}-\boldsymbol{B}_{int})\,\mathrm{d}\,\Omega + \int_{4\pi} B_r\boldsymbol{\xi}\cdot\delta\boldsymbol{B}_{int}\,\mathrm{d}\,\Omega\right] = -\Omega^2 E_0. \tag{1.45}$$

Considering equations 1.41, (1.44) and (1.45), the internal energy perturbation due to the surface curents is found to be

$$\boxed{\delta W_{sc} = \Omega^2 E_0(A-1)}\,, \tag{1.46}$$

which agrees with (1.27) up to order Ω^2 as expected.

The ambiguity in (1.40) can also be solved in another way, which doesnt require the radial component of the displacement field to vanish at the surface. In the equilibrium configuration, it is unlikely that surface currents are present, because they would dissipate rapidly. Besides, strong surface currents would be required to produce an important discontinuity in $\boldsymbol{B}$, which is unlikely to happen because the fluid density decreases drastically near the surface. This means that not only the r component of the field must be continuous, but rather that the equilibrium equilibrium magnetic field must be completely continuous along the boundary. This does not eliminate the ambiguity in the selection of $\delta\boldsymbol{B}$, but it does not matter because the first term in δW_{sc} is zero.

Chapter 2

Smooth Flowers & Ruderman's instability

Now I consider the effects of performing the cut of the star smoothly across a region of finite width $2\epsilon R$. The motivation behind this is that when a toroidal field is added in order to stabilize the star, the flux through the plane that cuts it in half in Flowers & Ruderman's instability is no longer zero. Thus, if a sharp cut is done, magnetic field lines would be cut, which is not possible. Because of this, an arbitrarily weak toroidal field is enough to stabilize the star against the sharp cut, but if the cut is done smoothly as described above, toroidal field lines will not be cut, but instead will be severely twisted. As ϵ increases, this bending will be less pronounced, and thus the stabilizing effect of the toroidal field will be reduced. Under some reasonable assumptions, I use perturbation theory to obtain a ratio between the energy of the poloidal field and the total energy of the magnetic field for which the field becomes stable to this displacement. This value can be compared with the values obtained by Braithwaite (2009) for which the field becomes unstable.

To do this, I consider a displacement field of the form

$$\boldsymbol{\xi} = \begin{cases} -\Omega_0 r \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} & x < -\epsilon R \\ \Omega(x) r \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} & |x| < \epsilon R \\ \Omega_0 r \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} & x > \epsilon R \end{cases} \tag{2.1}$$

where $\Omega(x)$ is a continuous, **odd** function in the interval $|x| < \epsilon R$ that satthatisfies

$$\Omega(\pm\epsilon R) = \pm\, \Omega_0, \quad \frac{\mathrm{d}\,\Omega}{\mathrm{d}\,x}_{x=\pm\epsilon R} = \quad 0. \tag{2.2}$$

The condition imposed on the derivative is to avoid discontinuities in $\delta\boldsymbol{B}$ along the boundary, which would in turn produce perturbed surface currents. Similar to the displacement field used before for the sharp cut, $\boldsymbol{\xi}$ has no $\hat{\boldsymbol{r}}$ component, as expected from a stably stratified star, and it satisfies $\nabla \cdot \boldsymbol{\xi} = 0$, so there will be no fluid contribution to δW.

The internal energy perturbation for this displacement field can be splitted into several terms, including a term that involves surface currents, δW_{sc}. This contribution to the internal energy perturbation involves surface integrals of an infinite number of spherical harmonics, and the fact that the displacement field is defined in terms of cartesian coordinates adds great complexity in trying to evaluate δW_{sc}. Because of this, I consider that the smooth transition is done in a thin region relative to the radius of the star, so $\epsilon \ll 1$, and I assume that δW_{sc} does not change significantly with respect to the value obtained for the sharp cut[1]. In any case, we expect δW_{sc} to increase as ϵ increases, since in this case the dipole component of the external magnetic field will not be reduced as much as was the case for the sharp cut.

2.1 Cylinder approximation and toroidal fields

As a simple approximation to the region of transition ($|x| < \epsilon R$), I will consider it as a cylinder of height $2\epsilon R$ and radius R. The coordinates in this system will be ϖ for the cylindrical radial coordinate,

[1] I do not expect the external magnetic field to be significantly different on the surface of the star for the region $|x| > \epsilon R$, so the contribution to δW_{sc} on this region should not change significantly. Also, the area of the surface in the region $|x| < \epsilon R$ is small compared to the rest of the surface in which the integral for δW_{sc} is done, so even if there are significant changes there, I do not expect them to significantly modify the work done on the whole surface.

z' oriented in such a way that $\hat{\boldsymbol{z}}'$ coincides with the previous Cartesian $\hat{\boldsymbol{x}}$, and the azimuthal angle ϑ in such a way that $\vartheta = 0, z' = 0$ is equivalent to the previous Cartesian z axis. The direction of increasing ϑ is chosen in such a way that the basis vectors for the cylindrical coordinate system satisfy (as expected) that $\hat{\boldsymbol{\varpi}} \times \hat{\boldsymbol{\vartheta}} = \hat{\boldsymbol{z}}'$.

The displacement field in this region can be written as

$$\boldsymbol{\xi} = \varpi\Omega(z')\hat{\boldsymbol{\vartheta}}. \tag{2.3}$$

I consider the perturbation in the potential energy of a toroidal field due to this displacement. Since the height of the cylinder is small relative to the radius of the star, I approximate the field as

$$\boldsymbol{B} = g(\varpi, \vartheta)\hat{\boldsymbol{z}}' \tag{2.4}$$

where $g(\varpi, \vartheta)$ is a 2π-periodic function that is odd in θ. Using this, the internal energy perturbation in this region is

$$\delta W_T = -\frac{1}{8\pi}\int_{-\epsilon R}^{\epsilon R} \mathrm{d}\, z' \int_0^{2\pi} \mathrm{d}\,\vartheta \int_0^R \mathrm{d}\,\varpi \left[\Omega^2 \left\{\left(\frac{\partial g}{\partial \vartheta}\right)^2 + g\frac{\partial^2 g}{\partial \vartheta^2}\right\} + \varpi^2 g^2 \Omega \frac{\mathrm{d}^2\Omega}{\mathrm{d}\, z'^2}\right]\varpi. \tag{2.5}$$

This expression can be simplified by noting that $\left(\frac{\partial g}{\partial \vartheta}\right)^2 + g\frac{\partial^2 g}{\partial \vartheta^2} = \frac{\partial}{\partial \vartheta}\left(g\frac{\partial g}{\partial \vartheta}\right)$. Because of this, the integral over ϑ of this term is be equal to zero, and

$$\delta W_T = -\frac{1}{8\pi}\int_{-\epsilon R}^{\epsilon R} \mathrm{d}\, z'\, \Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}\, z'^2}\int_0^{2\pi} \mathrm{d}\,\vartheta \int_0^R \mathrm{d}\,\varpi\; \varpi^3 g^2. \tag{2.6}$$

Furthermore, since I demand that the derivative of Ω vanishes on the boundary, this can be rewritten as

$$\delta W_T = \frac{1}{8\pi}\int_{-\epsilon R}^{\epsilon R} \mathrm{d}\, z' \left(\frac{\mathrm{d}\,\Omega}{\mathrm{d}\, z'}\right)^2 \int_0^{2\pi} \mathrm{d}\,\vartheta \int_0^R \mathrm{d}\,\varpi\varpi^3 g^2. \tag{2.7}$$

From this, it can be seen immediately that $\delta W_T > 0$, so, as expected, the toroidal field opposes this displacement. From this point, not too much can be done but to specify a model for both $\Omega(z')$ and $g(\varpi, \vartheta)$.

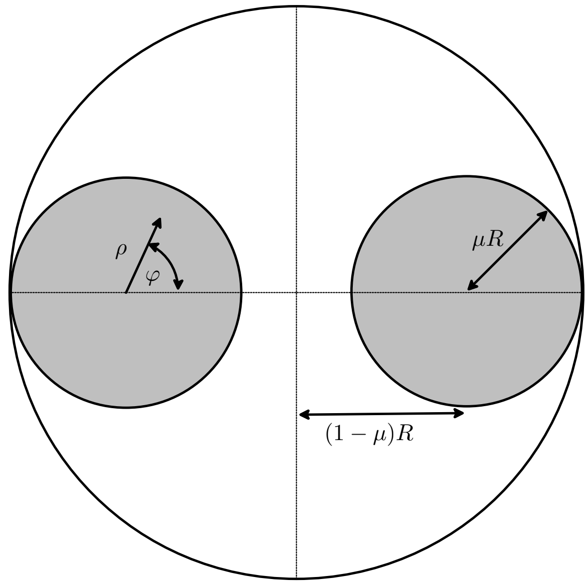

Figure 2.1: Model used for the toroidal field. The vertical line is the symmetry axis, and the field is contained in a circular torus of radius μR that is tangent to the equator of the star, as shown by the shaded region in the figure. Also depicted in the figure are the coordinates ρ and φ used to describe the magnitude of the field. In the cylinder approximation the torus is treated as two independent cylindrical regions.

I choose my function $\Omega(z')$ as

$$\Omega(z') = \Omega_0 \sin\left(\frac{z'\pi}{2\epsilon R}\right) \quad \Rightarrow \quad \delta W_T = \frac{\pi \Omega_0^2}{32\epsilon R}\int_0^{2\pi} \mathrm{d}\,\theta \int_0^R \mathrm{d}\,\varpi\,\varpi^3 g^2. \tag{2.8}$$

This function $\Omega(z')$ is odd and satisfies the required conditions mentioned in equation (2.2). With this particular displacement field, $\delta W_T \propto \epsilon^{-1}$. So, as mentioned before, if the region where the displacement field switches direction is very thin, the magnetic energy will increase significantly, and thus an infinitely weak toroidal field is enough to stabilize the star against a sharp cut.

Since the toroidal field is confined within the poloidal field lines that are closed inside the star[2], I consider the toroidal field to be contained in a torus of internal radius μR. In the cylinder approximation, I consider this torus as two cylindrical regions of radius μR that are centered at $(\varpi, \vartheta) = (R(1-\mu), \pm\pi/2)$, as ilustrated in figure 2.1. In each of these regions, the strength of the field will depend on the distance

[2]This is required for the star to be in an axisymmetric equilibrium, since if a toroidal field is present outside this region, the magnetic field produces forces in the azimuthal direction that cannot be countered by any hydrostatic effect.

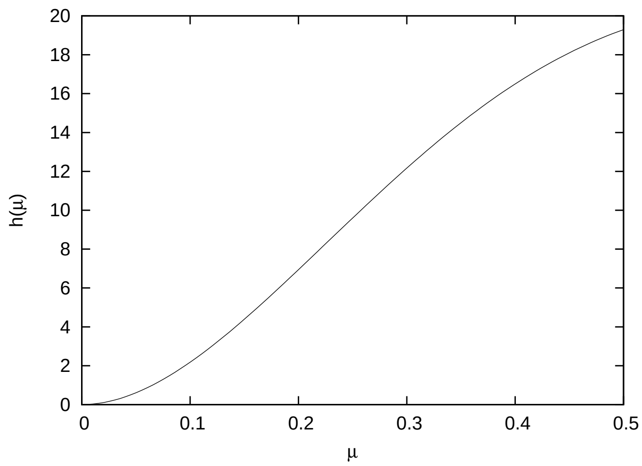

Figure 2.2: Plot of the function $h(\mu)$ from equation (2.11)

to the center of the cylinder[3], so I switch to coordinates (ρ, φ) centered on one of these circles in which I have $g = g(\rho)$ (as is shown in figure 2.1). The corresponding δW_T can be solved in these coordinates as

$$\delta W_T = \frac{\pi \Omega_0^2}{32 \epsilon R} \int_0^{2\pi} \mathrm{d}\,\vartheta \int_0^R \mathrm{d}\,\varpi\; \varpi^3 g(\varpi, \vartheta)^2 = \frac{\pi \Omega_0^2}{16 \epsilon R} \int_0^{2\pi} \mathrm{d}\,\varphi \int_0^{\mu R} \mathrm{d}\,\rho\; \rho g(\rho)^2 (d(\rho, \varphi))^2, \tag{2.9}$$

where $(d(\rho, \varphi))^2 = \rho^2 + [R(1-\mu)]^2 - 2\rho R(1-\mu)\cos\varphi$ is the distance to the origin. As a model for $g(\rho)$, I use

$$g(\rho) = \eta B_P \cos^2\left(\frac{\rho\pi}{2\mu R}\right), \tag{2.10}$$

where B_P is the maximum strength of the poloidal field on the surface, and ηB_P is the maximum strength of the toroidal field. The square on the cosine is necessary for $\delta \boldsymbol{B}$ to be continuous along the surface where the toroidal field vanishes. Using this model for the field, δW_T results in

$$\boxed{\delta W_T = \frac{3\Omega_0^2 \eta^2 E_0}{64 \epsilon \pi^2} h(\mu), \qquad h(\mu) = (9\pi^4 - 77\pi^2 + 192)\mu^4 + \mu^2\pi^2(6\pi^2 - 32)(1 - 2\mu)}\,, \tag{2.11}$$

where E_0 is the initial energy of the exterior magnetic field. A plot of the function $h(\mu)$ is shown in

[3]It can be seen from (2.9) that the detailed geometry of the toroidal field is not so relevant, specially if the toroidal field is contained in a region far away from the center of the star. In the latter case, $d(\rho, \varphi) \sim R(1-\mu)$, and the integral will involve only the square of the magnitude of the magnetic field times an area element. Because of this, δW_T should be closely related to the energy of the magnetic field, rather than its detailed geometry.

figure 2.2, and it can be seen that δW_T increases with μ.

2.2 Effect of poloidal fields for the smooth rotation

2.2.1 Cross term in δW

When a poloidal field is added, a cross term appears in δW that involves both the poloidal and toroidal components of the magnetic field. This term has the form

$$\delta W_{cross} = -\frac{1}{2}\int_V \mathrm{d}\,V \boldsymbol{\xi}\cdot\left[\delta\boldsymbol{j}_T\times\boldsymbol{B}_P+\boldsymbol{j}_T\times\delta\boldsymbol{B}_P+\delta\boldsymbol{j}_P\times\boldsymbol{B}_T+\boldsymbol{j}_P\times\delta\boldsymbol{B}_T\right]. \tag{2.12}$$

where $\boldsymbol{j}_P$ and $\boldsymbol{j}_T$ are the currents related to the poloidal and toroidal fields respectively[4], so

$$4\pi\boldsymbol{j}_P = \nabla\times\boldsymbol{B}_P, \qquad 4\pi\boldsymbol{j}_T = \nabla\times\boldsymbol{B}_T. \tag{2.13}$$

I consider this term in Cartesian coordinates, requiring only that the magnetic field be axisymmetric, without specifying the actual configuration of the toroidal and poloidal components of the magnetic field. The displacement field is as in equation (2.1), with

$$\Omega(x) = \sin\left(\frac{x\pi}{2\epsilon}\right). \tag{2.14}$$

Considering only the parity of the functions involved, it can be proved that the integrand in δW_{cross} is an odd function of x, and since the integral is over a sphere, integration over x will immediately give zero as a final result, so

$$\boxed{\delta W_{cross} = 0}. \tag{2.15}$$

The detail on how the parity of the integrand is obtained can be found in appendix A.

2.2.2 Purely poloidal contribution to δW

Using the cylinder approximation it is difficult to treat the contribution to δW due only to the poloidal field. It is also difficult to treat the problem in spherical coordinates, since the regions of

[4] Because of this, $\boldsymbol{j}_P$ is actually a toroidal field and $\boldsymbol{j}_T$ is a poloidal field.

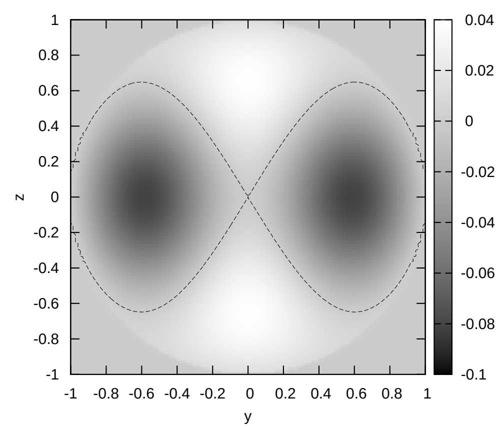

Figure 2.3: Integrand for δW_P, $-\boldsymbol{\xi}\cdot(\delta\boldsymbol{j}_P\times\boldsymbol{B}_P+\boldsymbol{j}_P\times\delta\boldsymbol{B}_P)/2$, in the plane given by $x=\epsilon/2$ using $\epsilon=1/10$, with the line where the integrand is equal to zero plotted on top.

integration involved are non-trivial. However, for certain particular choices of the poloidal field, the purely poloidal contribution to δW can be solved exactly using Cartesian coordinates. The displacement field considered here is the same one that was used in §2.2.1. I use a dipole field equivalent to the one used by Akgün (2009 private communication), but normalized so at the surface the maximum strength is B_P. The function $f(r)$ for this field is[5]

$$f(r)=\frac{35B_p}{16}\left[r^2-\frac{6}{5}\frac{r^4}{R^2}+\frac{3}{7}\frac{r^6}{R^4}\right]. \tag{2.16}$$

Solving for the contribution to δW due purely to the poloidal field, one obtains a finite polynomial in ϵ that to lowest order is

$$\delta W_P=\frac{(23\pi^2-330)}{8192}B_P^2R^3\Omega_0^2\epsilon=\frac{(69\pi^2-990)}{2048}E_0\Omega_0^2\epsilon\simeq-0.15\epsilon E_0\Omega_0^2. \tag{2.17}$$

This contribution is negative, but it is not as important as that of δW_{sc} (from equation (1.46) it can be seen that $\delta W_{sc}\simeq-0.57\Omega^2E_0$). Initially we expected the poloidal field to perform a stabilizing effect,

[5]This field is completely continuous along the surface of the star, so there are no surface currents present in the equilibrium configuration. Also, it satisfies $|\boldsymbol{j}|=0$ at the surface, which is expected from the fact that the matter density goes to zero there.

since this displacement would tend to twist field lines that are near to the symmetry axis. However, the region where the poloidal field lines are closed within the star turns out to be highly unstable to this displacement, as can be seen in figure 2.3. It can be seen that the contribution to the internal energy is positive along the axis of symmetry, and the region where it is negative encloses the field lines that are closed inside the star. We believe the positive contribution to be caused by the twisting of field lines, and the negative contribution to be due to an effect similar to that described by Markey & Tayler (1973) and Wright (1973).

2.3 Total internal energy perturbation

To obtain the total energy perturbation, I add all the contributions obtained so far,

$$\begin{aligned} \delta W &= \delta W_{sc} + \delta W_T + \delta W_P &\quad (2.18)\\ &= \Omega_0^2 E_0 \left[\frac{3\eta^2}{16\epsilon\pi^2} h(\mu) + \frac{69\pi^2 - 990}{2048}\epsilon + (A-1) \right]. &\quad (2.19) \end{aligned}$$

If $\delta W_t = 0$, then the system is marginally stable, and for that case, solving η^2 in terms of μ and ϵ results in

$$\eta^2 = \frac{\epsilon\pi^2}{384 h(\mu)} \left[2048(1-A) + 33(16 - 3\pi^2)\epsilon \right]. \quad (2.20)$$

Choosing μ and ϵ, we obtain from this a lower bound on the strength of the toroidal field needed to stabilize the star against a smooth rotation done on a region of width $2\epsilon R$. However, μ is not completely arbitrary, since in equilibrium, the toroidal field must be contained by the field lines that are closed inside the star. A reasonable value for μ (for the poloidal field chosen) is $\mu = 0.2$, which gives me $h(\mu) \sim 6.94$. Now, evaluating η in the above expression for $\epsilon = 1/3$ (which should be far above the region where this approximation is valid, and should serve as a good lower bound on the strength needed for the toroidal field), one obtains $\eta \sim 0.95$.

2.4 Comparing the poloidal and toroidal energy of the magnetic field

In order to compare this result with that of Braithwaite (2009), we must see what my result means in terms of the energies of the toroidal and poloidal fields. These energies can be evaluated as (the energy of the poloidal field includes also the external energy of the magnetic field)

$$E_P = \frac{35}{66} B_P^2 R^3 = \frac{70}{11} E_0, \tag{2.21}$$

$$E_T = \frac{B_p^2 R^3 \eta^2}{32\pi} \left(3\pi^2 - 16\right) \mu^2 (1-\mu) = \frac{3\eta^2}{8\pi} \left(3\pi^2 - 16\right) \mu^2 (1-\mu) E_0, \tag{2.22}$$

with this, the ratio of poloidal to total energy is

$$\frac{E_P}{E} = \frac{E_P}{E_T + E_P} = \frac{560\pi}{33(3\pi^2-16)\eta^2\mu^2(1-\mu) + 560\pi}. \tag{2.23}$$

For the values obtained in the previous section, I get a value of this ratio very close to unity, $E_P/E \sim 0.993$. This tells me that a toroidal field with an energy much smaller than the poloidal field is enough to stabilize the star against this perturbation. This can be compared with the instability that could be seen in the simulations by Braithwaite & Nordlund (2006) for a ratio of $E_P/E = 0.8$. As this perturbation happens with a much stronger toroidal field, all seems to indicate that the perturbation we are studying is not the dominant one, since other instabilities are present for the poloidal field even when the toroidal field is strong enough to stabilize it against the one we have studied.

Chapter 3

Conclusions and discussion

Flowers & Ruderman (1977) presented an argument that shows how purely poloidal fields in stars are unstable. If the external field is similar to a dipole, one could cut the star in half and rotate each piece in opposite directions, leading to a configuration in which the external field resembles a quadropole, and thus, the energy of the external magnetic field should be significantly reduced. This argument was given as an analogy to the case of two aligned magnets, in which case the antiparallel configuration has less energy. Although Flowers & Ruderman's instability mechanism is widely accepted, no formal proof had been given that shows both that the external magnetic energy is reduced when the rotation of each half is completed, and that the energy reduces monotonously along the entire process.

In this report I present a formal proof of this mechanism for the case in which the field outside the star is that of a point dipole, by solving the energy of the external field along the entire rotation. I showed that the external magnetic energy decreases monotonously, having a final value of approximately $0.57E_0$ where E_0 is the initial energy. When I proved that the final energy was less than the initial one I only required a quantity Υ (as defined by equation (1.13)) to be conserved when the perturbation is done, and Flowers and Ruderman's instability is just a particular case that satisfies this condition. Perhaps by studying what other displacements conserve Υ, other interesting instabilities responsible of reducing the external magnetic energy could be found.

I also studied Flowers and Ruderman's instability using perturbation theory, in which case I had to consider the effects of surface currents and perturbed surface currents in order for the instability to appear. These effects are not unique to Flowers & Ruderman's instability, and should be considered for any displacement that modifies the magnetic field on the surface. The result obtained for the internal energy perturbation of the star was found to be consistent with the exact value of the energy previously found.

I then studied how a toroidal field could stabilize the star against Flowers & Ruderman's instability. Since a sharp cut through the star would split toroidal field lines, the displacement has to be carried out with a continuous displacement field that switches the orientation of rotation across a thin region. For a specific model, it was found that the configuration was stable against Flowers & Ruderman's instability for a ratio of energies of the poloidal magnetic field to the total magnetic energy of $E_P/E < 0.993$. Using MHD simulations, Braithwaite (2009) had shown that when the ratio E_P/E was below 0.8, the instabilities driven by the poloidal field were supressed, but if the ratio was just above 0.8, the field was found to be unstable with an $m = 2$ mode that does not resemble Flowers & Ruderman's instability. Because of this, we conclude that Flowers & Ruderman's instability mechanism is not the dominant one.

I am currently constructing a displacement field that allows me to analytically reproduce the results in Braithwaite (2009), using the configurations for the toroidal and poloidal magnetic fields given in Chapter 2. The main feature that I expect to reproduce, is that the $m = 2$ mode becomes unstable for a ratio $E_P/E \simeq 0.8$ or higher, and from there, study how this critical value depends on the geometry of the toroidal and poloidal components of the field.

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Appendix A

Parity of the integrand in δW_{cross}

In this appendix, I prove that the integrand of δW_{cross} (that can be seen in (2.12)) is an odd function of x, and thus, $\delta W_{cross} = 0$. In order to do so, I need only to consider the parity of the functions involved. For simplicity, I now present a notation that allows one to easily solve the symmetries of complex expressions. The actual proof will be much more simple and evident in this notation. To begin with, I will use the following symbols to denote a function only by its parity,

$$e_x = \text{Even function of x}, \quad o_x = \text{Odd function of x}, \quad n_x = \text{Function with unknown parity in x.} \tag{A.1}$$

From these definitions, the following properties can be obtained direcly

- $e_x + e_x = e_x, \quad o_x + o_x = o_x, \quad e_x + o_x = n_x$
- $e_x \cdot e_x = e_x, \quad o_x \cdot o_x = e_x, \quad e_x \cdot o_x = o_x$
- $-e_x = e_x, \quad -o_x = o_x.$

If the functions we are considering are functions of the Cartesian coordinates[1] x, y and z, then the following properties can also be shown to be true

- $\partial_x e_x = o_x, \quad \partial_y e_x = \partial_z e_x = e_x$

[1]Perhaps more general results can be obtained when considering orthogonal coordinate systems, but only the results in cartesian coordinates are required.

- $\partial_x o_x = e_x, \quad \partial_y o_x = \partial_z o_x = o_x.$

I can also define vector fields considering only the parity in each of their components. For instance $\boldsymbol{v} = e_x\hat{\boldsymbol{x}} + e_x\hat{\boldsymbol{y}} + o_x\hat{\boldsymbol{z}}$ is a vector field whose x and y components are even functions of x, and whose z component is an odd function of x. A completely arbitrary vector field can be written as $\boldsymbol{n}_x = n_x\hat{\boldsymbol{x}} + n_x\hat{\boldsymbol{y}} + n_x\hat{\boldsymbol{z}}$. Two vector fields are of special interest, and I will denote them in the following way

$$\boldsymbol{e}_x = e_x\hat{\boldsymbol{x}} + o_x\hat{\boldsymbol{y}} + o_x\hat{\boldsymbol{z}}, \quad \boldsymbol{o}_x = o_x\hat{\boldsymbol{x}} + e_x\hat{\boldsymbol{y}} + e_x\hat{\boldsymbol{z}}, \tag{A.2}$$

so, these symbols $\boldsymbol{e}_x$ and $\boldsymbol{o}_x$ represent vector fields with similar parity in their y and z components, and opposite parity in its x component. From the properties given for e_x and o_x, the following properties can be proved:

- $\boldsymbol{e}_x \times \boldsymbol{e}_x = \boldsymbol{e}_x, \quad \boldsymbol{o}_x \times \boldsymbol{o}_x = \boldsymbol{e}_x, \quad \boldsymbol{e}_x \times \boldsymbol{o}_x = \boldsymbol{o}_x$
- $\boldsymbol{e}_x \cdot \boldsymbol{e}_x = e_x, \quad \boldsymbol{o}_x \cdot \boldsymbol{o}_x = e_x, \quad \boldsymbol{e}_x \cdot \boldsymbol{o}_x = o_x.$

Therefore, cross and inner products between these two kinds of vectors behave in a very similar way to the products of single functions. The last things required to complete the proof, are expressions for the curl and the gradient of these vectors. The corresponding properties of these differential operators are

- $\nabla \cdot \boldsymbol{e}_x = o_x, \quad \nabla \cdot \boldsymbol{o}_x = e_x$
- $\nabla \times \boldsymbol{e}_x = \boldsymbol{o}_x, \quad \nabla \times \boldsymbol{o}_x = \boldsymbol{e}_x.$

Now, using this notation, I can prove that δW_{cross} is an odd function of x. To recall, the integrand is

$$\varepsilon = \boldsymbol{\xi} \cdot [\delta\boldsymbol{j}_T \times \boldsymbol{B}_P + \boldsymbol{j}_T \times \delta\boldsymbol{B}_P + \delta\boldsymbol{j}_P \times \boldsymbol{B}_T + \boldsymbol{j}_P \times \delta\boldsymbol{B}_T]$$

where

$$\delta\boldsymbol{B}_P = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_P), \quad \delta\boldsymbol{B}_T = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_T), \quad 4\pi\delta\boldsymbol{j}_P = \nabla \times (\delta\boldsymbol{B}_P), \quad 4\pi\delta\boldsymbol{j}_T = \nabla \times (\delta\boldsymbol{B}_T). \tag{A.3}$$

And $\boldsymbol{\xi}$ has the form of (2.1). I now proceed to write $\boldsymbol{\xi}$, $\boldsymbol{B}_P$, and $\boldsymbol{B}_T$ as either $\boldsymbol{e}_x$ and $\boldsymbol{o}_x$. First, $\boldsymbol{\xi}$ has no $\hat{\boldsymbol{x}}$ component, and both its $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ components are odd in x. So, using the notation I just introduced, I can write $\boldsymbol{\xi}$ as $\boldsymbol{\xi} = o_x\hat{\boldsymbol{y}} + o_x\hat{\boldsymbol{z}}$. Since the x component of $\boldsymbol{\xi}$ is the zero function, I can consider it to be an even function of x, so

$$\boldsymbol{\xi} = e_x\hat{\boldsymbol{x}} + o_x\hat{\boldsymbol{y}} + o_x\hat{\boldsymbol{z}} = \boldsymbol{e}_x. \tag{A.4}$$

Now, $\boldsymbol{B}_T$ and $\boldsymbol{B}_P$ are axisymmetric, with $\hat{\boldsymbol{z}}$ being the axis of symmetry. $\boldsymbol{B}_T$ only has x and y components, with the x component being an even function of x, and the y component an odd function of x, so B_T can be written as $\boldsymbol{B}_T = e_x\hat{\boldsymbol{x}} + o_x\hat{\boldsymbol{y}}$. Just as before, since the z component is zero, I can choose it as o_x to get

$$\boldsymbol{B}_T = e_x\hat{\boldsymbol{x}} + o_x\hat{\boldsymbol{y}} + o_x\hat{\boldsymbol{z}} = \boldsymbol{e}_x \quad \Longrightarrow \quad \boldsymbol{j}_T = \nabla \times \boldsymbol{e}_x = \boldsymbol{o}_x. \tag{A.5}$$

B_P has components in x, y, and z, with its x component being an odd function of x, and its y and z components even functions of x,

$$\boldsymbol{B}_P = o_x\hat{\boldsymbol{x}} + e_x\hat{\boldsymbol{y}} + e_x\hat{\boldsymbol{z}} = \boldsymbol{o}_x \quad \Longrightarrow \quad \boldsymbol{j}_P = \nabla \times \boldsymbol{o}_x = \boldsymbol{e}_x. \tag{A.6}$$

The parity of the perturbed quantities can be solved by using the properties of $\boldsymbol{e}_x$ and $\boldsymbol{o}_x$ shown above,

$$\begin{aligned} \delta\boldsymbol{B}_T &= \nabla \times (\boldsymbol{e}_x \times \boldsymbol{e}_x) = \boldsymbol{o}_x, \quad & \delta\boldsymbol{B}_P &= \nabla \times (\boldsymbol{e}_x \times \boldsymbol{o}_x) = \boldsymbol{e}_x, \\ \delta\boldsymbol{j}_T &= \nabla \times (\boldsymbol{o}_x) = \boldsymbol{e}_x, & \delta\boldsymbol{j}_P &= \nabla \times (\boldsymbol{e}_x) = \boldsymbol{o}_x. \end{aligned} \tag{A.7}$$

The parity of the integrand is then solved to be

$$\varepsilon = \boldsymbol{e}_x \cdot [\boldsymbol{e}_x \times \boldsymbol{o}_x + \boldsymbol{o}_x \times \boldsymbol{e}_x + \boldsymbol{o}_x \times \boldsymbol{e}_x + \boldsymbol{e}_x \times \boldsymbol{o}_x] = o_x, \tag{A.8}$$

so, the integrand of δW_{cross} is an odd function of x, and thus, integrates to zero in the case we are interested in.